Mathematical Physics: Team Competition

(To complete this assignment, answer 2 out of the 3 problems.)

Problem 1

Consider a scalar field theory in D spacetime dimensions with global $O(N)$ symmetry and a quartic interaction potential. Let the scalar field transform either in the fundamental or adjoint representation of $O(N)$. The classical action is of the form:

$$S[\phi] = \int d^D x \left[\frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{\lambda}{4} \left(\phi^a \phi^a - v^2 \right)^2 \right], \quad a = 1, \dots, N.$$

1. Explain how the global $O(N)$ symmetry can be spontaneously broken when the scalar field transforms in the fundamental representation of $O(N)$. Determine the vacuum manifold and describe the pattern of symmetry breaking from $G = O(N)$ to a subgroup H.
2. Repeat the analysis assuming that the scalar field transforms in the adjoint representation of $O(N)$, by giving an example. Briefly discuss the non-uniqueness of the symmetry breaking pattern.
3. Explain the concept of non-linear realization of a spontaneously broken symmetry. Show how the scalar field ϕ^a in the fundamental representation can be expressed as exponential in the Goldstone fields π^m, such that the constraint $\phi^a \phi^a = v^2$ is satisfied. Discuss how elements of the unbroken subgroup H act linearly on π^m, while elements of G/H act non-linearly through the exponential map.
4. Write down the generic form of the effective action for the Goldstone modes at low energies. Explain the geometric meaning of the action and how the unbroken (H) and broken (G/H) symmetries leave it invariant.

Problem 2

Black Hole Thermodynamics from Microcanonical Ensemble Treat a Schwarzschild black hole as a statistical mechanical system with energy $E = Mc^2$ and entropy $S = \frac{k_B A}{4\ell_p^2}$.

1. Compute the density of states $\Omega(E)$ and show it grows exponentially.
2. Derive the negative heat capacity $C = -k_B \frac{E^2}{E_p^2}$, where E_p is the Planck energy.
3. Discuss the implications for black hole evaporation (Hawking radiation).

Problem 3 A non-relativistic spinless quantum particle of mass m is constrained to move in a spherical triangle $T \subset S^2$ where S^2 is a sphere of radius R with the usual $SO(3)$-invariant metric. The internal angles of the spherical triangle T are

$$\{\pi/2, \pi/2, \pi/3\}$$

There is no magnetic potential, $\vec{A} = 0$, and the potential vanishes $V \equiv 0$. Find the energy eigenvalues and their degeneracies.